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STRONG APPROXIMATION OF RENEWAL PROCESSES

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We develop a strong approximation of renewal processes. The consequences of this approximation are laws of the iterated logarithm and a Bahadur–Kiefer representation of the renewal process in terms of partial sums. The Bahadur–Kiefer representation implies that the rate of the strong approximation with the same Wiener process for both partial sums and renewal processes cannot be improved upon when the underlying random variables have finite fourth moments. We can generalize our results to the case of nonindependent and/or nonidentically distributed random variables.

strong approximation * Wiener process * renewal process

1. Introduction

Let $X, X_1, X_2, \dots$ be a sequence of independent identically distributed random variables satisfying the following conditions:

(i) $EX = \mu > 0$,

(ii) $0 < E(X-\mu)^2 = \sigma^2 < \infty$,

(iii) $E|X|^p < \infty$ for some $p > 2$.

Or, instead of (iii) we shall assume the stronger condition that

(iv) $E\exp(tX)$ exists in a neighbourhood of $t = 0$.

Put $S_0 = 0$, $S_n = \sum_{i=1}^n X_i$ and $M_n = \max(S_i, 1 \le i \le n)$. We consider the processes $L(t)$ and $N(t)$ on $[0, \infty)$, where

$$L(t) = \max(n \ge 0: M_n \le t), \qquad N(t) = \min(n > 0: S_n > t).$$

Clearly, $L(t) = N(t) + 1$, so it is enough to study the asymptotic properties of one of them. In this paper we will deal only with the process $N(t)$ which is known in the stochastic literature as the renewal process or as the first-passage time of the sequence $S_0, S_1, \dots$ from the interval $[0, t]$. The process $L(t)$ was introduced by

Heyde [7] as a generalized renewal process. Let

$$S_n(t)=\frac{1}{n}\sum_{i=1}^{[nt]} X_i/\mu, \quad 0\le t<\infty,$$

and let

$$\alpha_n(t)=\mu\sigma^{-1}n^{1/2}(S_n(t)-t), \quad 0\le t<\infty,$$

where $[x]$ denotes the integer part of x. If $N_n(t)$ denotes the generalized inverse of $S_n(t)$ defined by

$$N_n(t)=\inf\{u: S_n(u)>t\}, \quad 0\le t<\infty,$$

$(\inf\emptyset=\infty)$, then

$$N(nt\mu)=nN_n(t), \quad 0\le t<\infty. \tag{1.1}$$

The main idea of our paper is that $N_n(t)$ is the inverse of $S_n(t)$ and we can handle the process

$$\beta_n(t)=\mu\sigma^{-1}n^{1/2}(t-N_n(t))$$

easier than

$$Z_n(t)=\mu\sigma^{-1}(nt-N(nt\mu)).$$

By (1.1) we of course have that $n^{-1/2}Z_n(t)\equiv\beta_n(t)$.

The weak convergence of $n^{-1/2}Z_n(t)$ in Skorohod's space $\mathscr{D}[0,1]$ to a Wiener process was proved by Billingsley [4, Theorem 17.3], when X is assumed to be a positive random variable. Basu [2], Vervaat [17] and Gut [6] proved the weak convergence of $n^{-1/2}Z_n(t)$ under the conditions (i), (ii) only. Moreover, it follows from theorems of Iglehart and Whitt [8] and Vervaat [17] that the functional central limit theorems for renewal process and associated partial sums are equivalent. The proofs of these results are based on Donsker's well-known theorem (see for example Billingsley [4, p. 137]), which gives the weak convergence of $\alpha_n(t)$ to a Wiener process. We continue this work in the present paper and discuss the strong approximation of the renewal process as a consequence of the strong approximation of partial sums. We use the fundamental theorems of Komlós, Major and Tusnády [11–12] and Major [14] which are the best possible in the sense that no further restrictions on the distribution function of X can improve the rates in the almost sure approximation, unless X is already a normal random variable.

Theorem A. *If the underlying probability space $(\Omega,\mathscr{A},P)$ is rich enough we can define a Wiener process $\{W(t), t\ge 0\}$ on it such that*

$$P\left\{\max_{1\le k\le n}|\sigma^{-1}(S_k-k\mu)-W(k)|>C_1\log n\right\}\le D_1 n^{-\varepsilon} \tag{1.2}$$

for any $\varepsilon>0$, where $C_1=C_1(\varepsilon)$ and D_1 are positive constants, provided that conditions (ii) and (iv) are satisfied.

If we replace (iv) *by* (iii), *then we have*

$$P\left\{\max_{1\le k\le n}|\sigma^{-1}(S_k-k\mu)-W(k)|>n^{(1+\varepsilon)/p}\right\}\le D_2 n^{-\varepsilon} \tag{1.3}$$

for any $\varepsilon\in(0,\frac{1}{2}(p-2))$, *where* D_2 *is a positive constant and*

$$\lim_{n\to\infty} n^{-1/p}\max_{1\le k\le n}|\sigma^{-1}(S_k-k\mu)-W(k)|=0 \quad a.s. \tag{1.4}$$

Throughout this paper we assume that we are working on the probability space of Theorem A.

2. Strong approximation

The main result of this paper follows the pattern of Theorem A.

Theorem 2.1. (a) *If the conditions* (i), (ii) *and* (iv) *are satisfied, then*

$$P\left\{\sup_{0\le t\le 1}|Z_n(t)-W(nt)|>C_3 n^{1/4}(\log n)^{3/4}\right\}\le D_3 n^{-\varepsilon}$$

for any $\varepsilon>0$, *where* $C_3=C_3(\varepsilon)$ *and* D_3 *are positive constants.*

(b) *If the conditions* (i), (ii) *and* (iii) *are satisfied, then*

$$P\left\{\sup_{0\le t\le 1}|Z_n(t)-W(nt)|>C_4 n^{1/4}(\log n)^{3/4}\right\}\le D_4 n^{-\varepsilon}$$

for any $\varepsilon\in(0,\frac{1}{4}(p-4))$ $(p>4)$ *and*

$$P\left\{\sup_{0\le t\le 1}|Z_n(t)-W(nt)|>C_5 n^{(1+\varepsilon)/p}\right\}\le D_5 n^{-\varepsilon}$$

for any $\varepsilon\in(\max(0,\frac{1}{4}(p-4)),\frac{1}{2}(p-2))$, *where* $C_4=C_4(\varepsilon)$ *and* D_4 *are positive constants.*

(c) *If the conditions* (i), (ii) *and* (iii) *are satisfied and* $2<p<4$, *then*

$$\lim_{n\to\infty} n^{-1/p}\sup_{0\le t\le 1}|Z_n(t)-W(nt)|=0 \quad a.s.,$$

and if $p\ge 4$, *then*

$$\limsup_{n\to\infty}(n\log\log n)^{-1/4}(\log n)^{-1/2}\sup_{0\le t\le 1}|Z_n(t)-W(nt)|=2^{1/4}\mu^{-1/2}\sigma^{1/2} \quad a.s.$$

As we pointed out, $\beta_n(t)\equiv n^{-1/2}Z_n(t)$, so it is easy to formulate the analogue of this theorem for the process $\beta_n(t)$, or equivalently, to derive Theorem 2.1 if we prove the corresponding theorem for $\beta_n(t)$.

Before proving Theorem 2.1 we prove an easy but efficient and useful lemma on the inverse of step functions. A function φ, defined on $[0,\infty)$ is called a step function if there is a decomposition of $[0,\infty)=\bigcup_{i=1}^{\infty}[t_i, t_{i+1})$ such that $0=t_1<t_2<\cdots$ and $\varphi(t)=q_i$, $t_i\leq t<t_{i+1}$, where q_i, $i=1,2,\ldots$, is a sequence of real numbers. We assume that $q_1=0$. The inverse of φ is

$$\psi(u)=\inf\{t\geq 0\colon \varphi(t)>u\},\quad 0\leq u<\infty,$$

$(\inf\emptyset=\infty)$.

Lemma. *For any* $T\geq 0$,

$$\sup_{0\leq u\leq T}|\psi(u)-u|\leq \sup_{0\leq t\leq\psi(T)}|\varphi(t)-t|.$$

Proof. The inverse function ψ is also a step function. We assume that $\psi(T)<\infty$. Let $\hat{q}_k$ be the first point of discontinuity of ψ after T, $T<\hat{q}_k$ and let $\hat{q}_1,\ldots,\hat{q}_{k-1}$ denote the points of discontinuity of ψ before T ($k=1$ is possible). We have that

$$\sup_{0\leq u\leq T}|\psi(u)-u|\leq\max(\hat{t}_1, |\psi(\hat{q}_i)-\hat{q}_i|, |\psi(\hat{q}_i-)-\hat{q}_i|,$$
$$1\leq i\leq k-1, |\psi(\hat{q}_k-)-\hat{q}_k|),$$

where $\hat{t}_i$, $1\leq i\leq k$ is defined as $\hat{q}_i=\varphi(\hat{t}_i)$ and $\hat{t}_i$ are points of discontinuity of φ. We get that

$$|\psi(\hat{q}_i-)-\hat{q}_i|=|\varphi(\hat{t}_i)-\hat{t}_i|,\quad 1\leq i\leq k,$$

$$\hat{t}_1=\sup_{0\leq t\leq\hat{t}_1}|\varphi(t)-t|$$

and

$$|\psi(\hat{q}_i)-\hat{q}_i|\leq\max\{|\varphi(\hat{t}_i)-\hat{t}_i|, |\varphi(\hat{t}_{i+1}-)-\hat{t}_{i+1}|\},\quad 1\leq i\leq k-1.$$

On the other hand, $\psi(T)=\hat{t}_k$, so we have proved our Lemma. If $\psi(T)=\infty$, then $\varphi(t)\leq T$ and therefore $\sup_{0\leq t<\infty}|\varphi(t)-t|=\infty$.

Proof of Theorem 2.1. We divide the proof into steps. We prove only statement (a) in detail because the proofs of statements (b) and (c) are similar to that of the case (a). We shall only formulate the corresponding forms of the steps of the proof.

The first step immediately follows from the Lemma.

Step 1.

$$\sup_{0\leq t\leq\tau}|N_n(t)-t|\leq\sup_{0\leq t\leq N_n(\tau)}|S_n(t)-t|.$$

Step 2. Let $\delta>0$. If $n\geq n_1$ is so large that

$$\mu\delta\sigma^{-1}(1+\delta)^{-1/2}n^{1/2}\geq\mu\sigma^{-1}(1+\delta)^{-1/2}n^{-1/2}+(2\varepsilon\log[n(1+\delta)])^{1/2}$$
$$+C_1[n(1+\delta)]^{-1/2}\log[n(1+\delta)],$$

then we have

$$p_{1n}=P\{N_n(1)>1+\delta\}\leq(D_1+1)[n(1+\delta)]^{-\varepsilon}.$$

Indeed, it follows from the definition of $N_n(1)$ that

$$p_{1n}=P\{S_n(1+\delta)\le 1\}=P\left\{\sigma^{-1}\left(\sum_{i=1}^{[n(1+\delta)]}X_i-\mu[n(1+\delta)]\right)\right.$$
$$\left.\le -\mu\sigma^{-1}([n(1+\delta)]-n)\right\}.$$

Hence, using (1.2) and the condition of Step 2, we get that

$$\begin{aligned}p_{1n}&\le P\{W([n(1+\delta)])\le -\mu\sigma^{-1}([n(1+\delta)]-n)+C_1\log[n(1+\delta)]\}\\&\quad+D_1[n(1+\delta)]^{-\varepsilon}\\&\le D_1[n(1+\delta)]^{-\varepsilon}+P\{W(1)\le -(2\varepsilon\log[n(1+\delta)])^{1/2}\}\\&\le (D_1+1)[n(1+\delta)]^{-\varepsilon}.\end{aligned}$$

Step 1, Step 2, (1.2) and the form of the exact distribution of $\sup_{0\le t\le 1}|W(t)|$ (see Csörgő and Révész [5, Theorem 1.5.1]) imply the following step.

Step 3. If $n\ge n_1$, then

$$p_{2n}=P\left\{\sup_{0\le t\le 1}|N_n(t)-t|>h_1(n)\right\}\le (2D_1+5)[n(1+\delta)]^{-\varepsilon}$$

with

$$\begin{aligned}h_1(n)&=\mu^{-1}n^{-1}+\sigma C_1\mu^{-1}n^{-1}\log[n(1+\delta)]\\&\quad+\sigma\mu^{-1}n^{-1}(2\varepsilon[n(1+\delta)]\log[n(1+\delta)])^{1/2}.\end{aligned}$$

Indeed, by elementary computation,

$$\begin{aligned}p_{2n}&\le (D_1+1)[n(1+\delta)]^{-\varepsilon}+P\left\{\sup_{0\le t\le 1+\delta}|S_n(t)-t|>h_1(n)\right\}\\&\le (2D_1+1)[n(1+\delta)]^{-\varepsilon}\\&\quad+P\left\{\max_{1\le k\le [n(1+\delta)]}|W(k)|>(2\varepsilon[n(1+\delta)]\log[n(1+\delta)])^{1/2}\right\}.\end{aligned}$$

For any fixed $T>0$ we have (the scale transformation)

$$\{W(t),0\le t\le T\}\stackrel{\mathcal{D}}{=}\left\{T^{1/2}W\left(\frac{t}{T}\right),0\le t\le T\right\},$$

so we obtain that the last probability is not greater than

$$P\left\{\sup_{0\le t\le 1}|W(t)|>(2\varepsilon\log[n(1+\delta)])^{1/2}\right\}\le 4[n(1+\delta)]^{-\varepsilon}.$$

Now consider the following decomposition of $\beta_n(t)$:

$$\beta_n(t)=\alpha_n(N_n(t))-\mu\sigma^{-1}n^{1/2}(S_n(N_n(t))-t). \tag{2.1}$$

First we approximate the first term in (2.1) and then we show that the second term is almost surely less than the rate of the approximation. Put

$$r_1(n)=\mu\sigma^{-1}n^{-1/2}+n^{-1/2}C_1\log[n(1+\delta)]$$
$$+(3(\varepsilon+1)\log[n(1+\delta)])^{1/2}(h_1(n))^{1/2}.$$

Step 4. If $n \geqslant n_1$, then

$$p_{3n}=P\left\{\sup_{0\leqslant t\leqslant 1}|\alpha_n(N_n(t))-n^{-1/2}W(nt)|>r_1(n)\right\}$$
$$\leqslant(5D_1+7)[n(1+\delta)]^{-\varepsilon}+C(1)(1+\delta)(h_1(n))^{-1}[n(1+\delta)]^{-(\varepsilon+1)},$$

where $C(1)$ is a positive constant.

Using Step 1 and (1.2) we get that

$$p_{3n}\leqslant P\left\{\sup_{0\leqslant t\leqslant 1}|\alpha_n(N_n(t))-n^{-1/2}W(nN_n(t))|\right.$$
$$\left.>\mu\sigma^{-1}n^{-1/2}+n^{-1/2}C_1\log[n(1+\delta)]\right\}$$
$$+P\left\{\sup_{0\leqslant t\leqslant 1}|W(nN_n(t))-W(nt)|\right.$$
$$\left.>(3(\varepsilon+1)nh_1(n)\log[n(1+\delta)])^{1/2}\right\}.$$

We estimate the random increment of the Wiener process with the help of Steps 2, 3 and Lemma 1.2.1 in [5]. The latter probability is less than or equal to

$$P\left\{\sup_{0\leqslant t\leqslant n(1+\delta)-nh_1(n)}\ \sup_{0\leqslant s\leqslant nh_1(n)}|W(t+s)-W(t)|\right.$$
$$\left.>(3(\varepsilon+1)nh_1(n)\log[n(1+\delta)])^{1/2}\right\}+(3D_1+6)[n(1+\delta)]^{-\varepsilon}$$
$$\leqslant C(1)n(1+\delta)(nh_1(n))^{-1}[n(1+\delta)]^{-(\varepsilon+1)}+(3D_1+6)[n(1+\delta)]^{-\varepsilon},$$

where $C(1)$ is the constant of Lemma 1.2.1 in [5].

The last step in the proof of the statement (a) is based on an estimation of the increments of partial sums. Set

$$v_1(n)=2\mu\sigma^{-1}n^{-1/2}+2C_1n^{-1/2}\log[n(1+\delta)]$$
$$+n^{-1/2}(3(\varepsilon+1)\log[n(1+\delta)])^{1/2}.$$

Step 5. If $n \geq n_1$, then

$$p_{4n} = P\left\{ \sup_{0\leq t\leq 1} \mu\sigma^{-1} n^{1/2} |S_n(N_n(t)) - t| > v_1(n) \right\}$$

$$\leq (2D_1 + 1)[n(1+\delta)]^{-\varepsilon} + C(1) n(1+\delta)[n(1+\delta)]^{-\varepsilon}.$$

It follows from the definition of N_n and S_n that $\sup_{0\leq t\leq 1} |S_n(N_n(t)) - t|$ equals the largest jump of $|S_n(t) - t|$ on $[0, N_n(1)]$, whence

$$p_{4n} \leq P\left\{ \sup_{0\leq t\leq N_n(1) - n^{-1}} \sup_{0\leq s\leq n^{-1}} |\alpha_n(t+s) - \alpha_n(t)| > v_1(n) \right\}.$$

Again using Steps 2, 3 and (1.2), we obtain that

$$p_{4n} \leq (D_1 + 1)[n(1+\delta)]^{-\varepsilon}$$

$$+ P\left\{ \sup_{0\leq t\leq 1+\delta - n^{-1}} \sup_{0\leq s\leq n^{-1}} |\alpha_n(t+s) - \alpha_n(t)| > v_1(n) \right\}$$

$$\leq (2D_1 + 1)[n(1+\delta)]^{-\varepsilon}$$

$$+ P\left\{ \sup_{0\leq t\leq 1+\delta - n^{-1}} \sup_{0\leq s\leq n^{-1}} n^{-1/2} |W(n(t+s)) - W(nt)| \right.$$

$$\left. > n^{-1/2} (3(\varepsilon+1) \log[n(1+\delta)])^{1/2} \right\}.$$

As was done in Step 4, the last probability can easily be estimated by Lemma 1.2.1 in [5]. It is

$$P\left\{ \sup_{0\leq t\leq n(1+\delta)-1} \sup_{0\leq s\leq 1} |W(t+s) - W(t)| > (3(\varepsilon+1) \log[n(1+\delta)])^{1/2} \right\}$$

$$\leq C(1) n(1+\delta)[n(1+\delta)]^{-(\varepsilon+1)}.$$

For the orders of $h_1(n)$, $r_1(n)$ and $v_1(n)$ we have

$$h_1(n) = O(n^{-1/2} (\log n)^{1/2}), \qquad r_1(n) = O(n^{-1/4} (\log n)^{3/4})$$

and

$$v_1(n) = O(n^{-1/2} \log n).$$

Since it is clearly enough to establish the theorem for large enough n, statement (a) of the theorem immediately follows from Step 4 and 5.

In the proof of (b) we have to replace (1.2) by (1.3) and in this way we get the following steps.

Step 2'. Let $\delta > 0$. If $n \geq n_2$ is so large that

$$\mu\delta\sigma^{-1}(1+\delta)^{-1/2} n^{1/2} \geq \mu\sigma^{-1}(1+\delta)^{-1/2} n^{-1/2} + (2\varepsilon \log[n(1+\delta)])^{1/2}$$

$$+ [n(1+\delta)]^{(1+\varepsilon)/p},$$

then

$$P\{N_n(1) > 1+\delta\} \leq (D_2+1)[n(1+\delta)]^{-\varepsilon}.$$

Step 3′. If $n \geq n_2$, then

$$P\left\{\sup_{0\leq t\leq 1} |N_n(t) - t| > h_2(n)\right\} \leq (2D_2+5)[n(1+\delta)]^{-\varepsilon},$$

where

$$h_2(n) = \mu^{-1}n^{-1} + \sigma\mu^{-1}[n(1+\delta)]^{(1+\varepsilon)/p}$$
$$+\sigma\mu^{-1}n^{-1}(2\varepsilon[n(1+\delta)]\log[n(1+\delta)])^{1/2}.$$

Step 4′. If $n \geq n_2$, then

$$P\left\{\sup_{0\leq t\leq 1} |\alpha_n(N_n(t)) - n^{-1/2}W(nt)| > r_2(n)\right\}$$
$$\leq (5D_2+7)[n(1+\delta)]^{-\varepsilon} + C(1)(1+\delta)(h_2(n))^{-1}[n(1+\delta)]^{-\varepsilon+1},$$

where

$$r_2(n) = \mu\sigma^{-1}n^{-1/2} + n^{-1/2}[n(1+\delta)]^{(1+\varepsilon)/p}$$
$$+(3(\varepsilon+1)h_2(n)\log[n(1+\delta)])^{1/2}.$$

Step 5′. If $n \geq n_2$, then

$$P\left\{\sup_{0\leq t\leq 1} \mu\sigma^{-1}n^{1/2}|S_n(N_n(t)) - t| > v_2(n)\right\}$$
$$\leq (2D_2+1)[n(1+\delta)]^{-\varepsilon} + C(1)n(1+\delta)[n(1+\delta)]^{-(\varepsilon+1)},$$

where

$$v_2(n) = 2\mu\sigma^{-1}n^{-1/2} + 2n^{-1/2}[n(1+\delta)]^{(1+\varepsilon)/p}$$
$$+n^{-1/2}(3(\varepsilon+1)\log[n(1+\delta)])^{1/2}.$$

To complete the proof of the statements in (b), it is enough to compute the orders of the rates:

$$h_2(n) = \mathrm{O}(\max(n^{-1+(1+\varepsilon)/p}, n^{-1/2}(\log n)^{1/2})),$$
$$r_2(n) = \mathrm{O}(\max(n^{-1/2+(1+\varepsilon)/2p}(\log n)^{1/2}, n^{-1/4}(\log n)^{3/4}))$$

and

$$v_2(n) = \mathrm{O}(n^{-1/2+(1+\varepsilon)/p}),$$

where $\varepsilon \in (0, \frac{1}{2}(p-2))$.

In this section we prove only half of (c), in the case $p \geq 4$, that is, only the inequality

$$\limsup_{n\to\infty} (n\log\log n)^{-1/4}(\log n)^{-1/2} \sup_{0\leq t\leq 1} |Z_n(t) - W(nt)|$$
$$\leq 2^{1/4}\mu^{-1/2}\sigma^{1/2} \quad \text{a.s.} \tag{2.2}$$

The opposite of (2.2) will be proved only after Theorem 3.2. In order to prove (2.2) we have to replace the probability inequalities in the corresponding steps by almost surely inequalities. Step 2′ implies that

$$\limsup_{n\to\infty} N_n(1) \leq 1 \quad \text{a.s.} \tag{2.3}$$

and therefore, using our Lemma, we can easily deduce from the law of iterated logarithm for partial sums that

$$\limsup_{n\to\infty} \left(\frac{n}{2\log\log n}\right)^{1/2} \sup_{0\leq t\leq 1} |N_n(t) - t| \leq \sigma\mu^{-1} \quad \text{a.s.} \tag{2.4}$$

Following the proof of Steps 4 and 5 we have to determine the exact order of the increments of the Wiener process. Let $\delta > 0$ and

$$h_3(n) = (1+\delta)\sigma\mu^{-1}(2n\log\log n)^{1/2}.$$

It was proved by Csörgő and Révész [5, Theorem 1.2.1] that

$$\limsup_{n\to\infty} (h_3(n)\log n)^{-1/2} \sup_{0\leq t\leq n(1+\delta)} \sup_{0\leq s\leq h_3(n)} |W(t+s) - W(t)| \leq 1 \quad \text{a.s.} \tag{2.5}$$

If we replace Steps 2, 3 with (2.3) and (2.4) and use (1.4) and (2.5) instead of the corresponding probability inequalities we can complete the proof of (2.2) and the proof of (c), in the case $2<p<4$.

3. Some consequences of the main theorem

The almost sure properties of the renewal process are the focus of this section. Using Theorem 2.1 we can determine the limit points of the process

$$\xi_n(t) = (N(nt\mu) - nt)(2\sigma^2\mu^{-2}n\log\log n)^{-1/2}.$$

This question was studied by Vervaat [17] who proved that $\xi_n(t)$ is a.s. relatively compact in $\mathscr{D}[0,\infty)$ under the conditions (i) and (ii). Let $\mathscr{S}$ be the set of absolutely continuous functions (with respect to Lebesgue measure) such that

$$f(0) = 0, \qquad \int_0^1 (f'(t))^2\,\mathrm{d}t \leq 1.$$

Theorem 3.1. *If the conditions* (i), (ii) *and* (iii) *are satisfied then the sequence* $\{\xi_n(t), 0\leq t\leq 1\}$ *is a.s. relatively compact with respect to the supremum norm, the set of its limit points is* $\mathscr{S}$,

$$\limsup_{n\to\infty} (n\log\log n)^{-1/2} \sup_{0\leq t\leq 1} |N(nt\mu) - nt| = 2^{1/2}\sigma\mu^{-1} \quad a.s.$$

and

$$\liminf_{n\to\infty}(n^{-1}\log\log n)^{1/2}\sup_{0\le t\le 1}|N(nt\mu)-nt| = 8^{-1/2}\pi\sigma\mu^{-1} \quad a.s.$$

Bahadur [1] was the first to investigate the distance between the empirical distribution function and its inverse, the empirical quantile function. Kiefer [10] determined the exact rate of this distance. The following theorem shows that a Bahadur–Kiefer type representation is also true for the renewal process in terms of partial sums and the rate of the distance is essentially the same as the rate of the distance between the empirical distribution and quantile function. Let

$$\Delta_n = \left|\mu N(nt\mu) + \sum_{i=1}^{[nt]} X_i - 2nt\mu\right|.$$

Theorem 3.2. *If the conditions* (i), (ii) *and* (iii) *are satisfied and* $2<p<4$ *then*

$$\lim_{n\to\infty} n^{-1/p}\Delta_n = 0 \quad a.s.,$$

and if $p\ge 4$ *then*

$$\limsup_{n\to\infty}(n\log\log n)^{-1/4}(\log n)^{-1/2}\Delta_n = 2^{1/4}\mu^{-1/2}\sigma^{3/2} \quad a.s.$$

Proof. The partial sums and the renewal process are approximated by the *same* Wiener process, so this theorem is an immediate consequence of Theorems A and 2.1, if $2<p<4$ and

$$\limsup_{n\to\infty}(n\log\log n)^{-1/4}(\log n)^{-1/2}\Delta_n \le 2^{1/4}\mu^{-1/2}\sigma^{3/2} \quad \text{a.s.},$$

if $p\ge 4$. If $p\ge 4$, then (2.1), and (1.4) show that

$$\limsup_{n\to\infty}(n\log\log n)^{-1/4}(\log n)^{-1/2}\Delta_n$$
$$= \sigma\limsup_{n\to\infty}(n\log\log n)^{-1/4}(\log n)^{-1/2}\sup_{0\le t\le 1}|W(N(nt\mu)-W(nt)| \quad \text{a.s.}$$

Consider the function $h_\delta(t)$ that equals t or $1-\delta$ according as $0\le t\le 1-\delta$, $1-\delta<t\le 1$, where $0<\delta<1$. Using part (iii) of Theorem 1.2.1 in [5] we get that

$$\lim_{n\to\infty}(n\log\log n)^{-1/4}(\log n)^{-1/2}$$
$$\times \sup_{1-\delta\le t\le 1}|W(nt+h_\delta(t)(2\sigma^2\mu^{-2}n\log\log n)^{1/2})-W(nt)|$$
$$=(1-\delta)^{1/2}2^{1/4}\mu^{-1/2}\sigma^{1/2} \quad \text{a.s.}$$

For each δ, $0<\delta<1$, $h_\delta\in\mathscr{S}$, so it follows from Theorem 3.1 that there is a sequence of random variables $n_k = n_k(\omega)$ such that

$$\lim_{k\to\infty} \sup_{0\le t\le 1} |(2\sigma^2\mu^{-2} n_k \log\log n_k)^{-1/2}(N(n_k t\mu) - n_k t) - h_\delta(t)| = 0 \quad \text{a.s.}$$

Since $\delta > 0$ is arbitrary, we have

$$\limsup_{n\to\infty} (n \log\log n)^{-1/4}(\log n)^{-1/2}\Delta_n \geq 2^{1/4}\mu^{-1/2}\sigma^{3/2} \quad \text{a.s.}$$

The Bahadur–Kiefer type representation immediately implies the opposite of (2.2) and we finished the proof of part (c) of Theorem 2.1.

With the help of Theorem 2.1 we can study the problems of how large and how small the increments of the renewal process over subintervals of length $h(n)$ of the interval $[0, n]$ can be when $n \to \infty$ and $h(n)$ is a nondecreasing function of n. Under conditions on order of $h(n)$ these increments are identical with ones of the Wiener process [5, Chapter 3].

A detailed characterization of the path behaviour of the renewal process can be given using the concept of upper–upper, upper–lower, lower–upper and lower–lower classes introduced by Lévy [13] and Révész [16]. Using the method of Jain, Jogdeo and Stout [9] we get the identity of some classes of $N(n\mu) - n$, $|N(n\mu) - n|$, $\sup_{0\le t\le 1}(N(nt\mu) - nt)$ and $\sup_{0\le t\le 1}|N(nt\mu) - nt|$ with the corresponding classes of the Wiener process. A bound for the rate of convergence of functionals (supremum, square-integral) of the renewal process can be deduced from Theorem 2.1.

Finally, we mention that using results of Philipp and Stout [15] and Berkes and Philipp [3], Theorems 2.1 and 3.2 can be partially extended to weakly dependent sequences. We shall return to these extensions and their applications in the approximation of processes of runs in a subsequent paper.

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References

[1] R.R. Bahadur, A note on quantiles in large samples, Ann. Math. Statist. 37 (1966) 577–580.
[2] A.K. Basu, Invariance theorems for first passage time random variables, Canad. Math. Bull. 15 (1972) 171–176.
[3] I. Berkes and W. Philipp, Approximation theorems for independent and weakly dependent random vectors, Ann. Probab. 7 (1979) 29–54.
[4] P. Billingsley, Convergence of Probability Measures (Wiley, New York, 1981).
[5] M. Csörgő and P. Révész, Strong Approximation in Probability and Statistics (Academic Press, New York, 1981).

[6] A. Gut, A functional central limit theorem connected with extended renewal theory, Z. Wahrsch. Verw. Gebiete 27 (1973) 123–129.
[7] C.C. Heyde, Asymptotic renewal results for a natural generalization of classical renewal theory, J. Roy. Statist. Soc. Ser. B. 29 (1967) 141–150.
[8] D.L. Iglehart and W. Whitt, The equivalence of functional central limit theorems for counting processes and associated partial sums, Ann. Math. Statist. 42 (1971) 1372–1378.
[9] N.C. Jain, K. Jogdeo and W.F. Stout, Upper and lower functions for martingales and mixing processes, Ann. Probab. 3 (1975) 119–145.
[10] J. Kiefer, Deviations between the sample quantile process and sample D.F. in: M.L. Puri ed., Nonparametric Techniques in Statistics (Cambridge University Press, London, 1970).
[11] J. Komlós, G. Tusnády and P. Major, An approximation of partial sums of independent R.V.'s and the sample D.F.I, Z. Wahrsch. Verw. Gebiete 32 (1975) 111–135.
[12] J. Komlós, G. Tusnády and P. Major, An approximation of partial sums of independent R.V.'s and the sample D.F.II, Z. Wahrsch. Verw. Gebiete 34 (1976) 33–58.
[13] P. Lévy, Processus Stochastique et Mouvement Brownien (Gauthier-Villars, Paris, 1948).
[14] P. Major, The approximation of partial sums of independent r.v.'s, Z. Wahrsch. Verw. Gebiete 35 (1976) 213–220.
[15] W. Philipp and W. Stout, Almost sure invariance principles for partial sums of weakly dependent random variables, Mem. Amer. Math. Soc. No. 161.
[16] P. Révész, On the increments of Wiener and related processes, Ann. Probab. 10 (1982) 613–622.
[17] W. Vervaat, Functional limit theorems for processes with positive drift and their inverses, Z. Wahrsch. Verw. Gebiete 23 (1972) 245–253.